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Nonuniqueness of a Massive Spin-2 Theory

J. B. NUNES

Departamento de Física, Universidade Federal Rural do Rio de Janeiro, Itaguai Rio de Janeiro, Brazil

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Abstract

The nonunique nature of massive spin-2 fields is explicitly shown in this paper through the construction of all possible field equations, using Dirac formalism for spin-$\frac{1}{2}$ fields. Out of these four possible theories, we point out two that do not show up scalar representations.

1. *Introduction*

I have tried elsewhere (Nunes, 1973), to construct a consistent massive spin-2 field theory, in a V_4 pseudo-Riemannian variety. This theory should represent the natural and standard version of a massive spin-2 field theory, previously formulated against the background of a Minkowskian space-time. However, even in flat space, there has been an apparent impossibility of building up a unique theory for such a field. In this paper we deliberately pick up Weyl spinors and use Dirac formalism, to make an explicit presentation of the four possible theories for massive spin-2 fields.

The difficulties we face in an F_{uv}-field theory, are twofold: (a) difficulties that are a common inheritance to higher spin fields; and (b) difficulties concerning the F_{uv} field itself, i.e., which are peculiar to it.

This paper is related to item (a), and, therefore, does not go beyond the simple assessment of the plain impossibility of a unique massive spin-2 field theory.

2. *Nonuniqueness of the Theory*

An important aspect of the theory of massive spin-2 fields, is the usual understanding that the mathematical entity suitable for describing such a field should be a trace-free symmetric second-rank tensor; this fact stems from the reducibility of the tensorial product $F^{vv} = F_A{}^v \otimes F_{B'}^v$, where $F_A{}^v$ and $F_B{}^v$ are self-representations of the homogeneous Lorentz group. It is well known that Lorentz groups possess tensorial and spinorial irreducible representations; this reality imposes a multiple choice of ways of setting up a massive spin-2 theory. As suggested by Dirac, if higher spin fields do exist, they could be described by a straightforward generalization of his equation for spin-$\frac{1}{2}$ fields, provided they are written using Weyl spinors. The fact that elementary particles should correspond to the irreducible representations of the inhomogeneous Lorentz group, is a corner-stone in field theory. From a pure group theoretical concept, if nothing rules out the existence of higher spin-2 fields, nonetheless, this statement leads invariably to a multiple number of theories.

Let us take Dirac's equation in Weyl spinors form. The spinors $\xi^A \zeta^A$ span a two-dimensional complex space, homomorphic to the self-representation space of the Lorentz group, whose representations are usually given in terms of two numbers j and j'; they identify the dimensionality of the spinorial representation space, whose dimension is $(2j + 1)(2j' + 1)$ and the product entities are now called spinors of order $2j + 2j'$.

$$\begin{aligned} \partial^{\dot{A}B}\xi_B &= \kappa\zeta A \\ \partial_{AB}\zeta^{\dot{A}} &= \kappa\xi_B \end{aligned} \tag{2.1}$$

The reason why this couple of numbers are needed lies in the fact that the spanning of the representation space is made not only using the two components of the spinor $\xi^A(\xi^1, \xi^2)$ but also its complex conjugate $\xi^A(\xi^1, \xi^2)$.

We shall now write down the formal generalization of (2.1):

$$\begin{aligned} \partial_{A\dot{B}}\xi^{AA_1A_2\cdots A_n}_{\dot{B}_1\dot{B}_2\cdots\dot{B}_m} &= \kappa\zeta^{A_1A_2\cdots A_n}_{\dot{B}\dot{B}_1\dot{B}_2\cdots\dot{B}_m} \\ \partial^{A\dot{B}}\zeta^{A_1A_2\cdots A_n}_{\dot{B}\dot{B}_1\cdots\dot{B}_m} &= \kappa\xi^{AA_1A_2\cdots A_n}_{\dot{B}_1\dot{B}_2\cdots\dot{B}_m} \end{aligned} \tag{2.2}$$

The fields these equations are meant to describe belong to higher spinorial representations of the Lorentz group, built up from the expansion basis $(\xi_A, \xi_{\dot{A}})$ or the set of components $(v_1, v_2, v_{\dot{1}}, v_{\dot{2}})$.

$$\theta_{rs,} = v_1^{r'-s'} v_{\dot{1}}^{r'-s'} v_2 s_{v_{\dot{2}}} s' \tag{2.3}$$

(these spinor products are made the same way we carry out the tensor product of the vectorial representation of Lorentz group).

Let $(r + 1)(r' + 1)$ be the dimensionality of the representation space. For the sake of convenience one usually finds in the literature r and r' expressed in the form $j = r/2; j' = r'/2$ so that $2j + 2j'$ becomes the order of the spinor. Let us call the space representation $\Lambda^{jj'}$. In terms of the number of dotted and undotted spinors in (2.2) we can express (jj') for ξ and ζ, respectively:

$$\begin{aligned} \xi &\to j_\xi = \tfrac{1}{2}(n + 1); j'_\xi = m/2 \\ \zeta &\to j'_\zeta = n/2; j'_\zeta = \tfrac{1}{2}(m + 1) \end{aligned}$$

If l is the highest spin found in the representation to which (ξ, ζ) belong, we have for each one of them

$$\xi \to l = j_\xi + j'_{\xi'} = \tfrac{1}{2}(m + n + 1)$$
$$\zeta \to l = j_\zeta + j'_\zeta = \tfrac{1}{2}(m + n + 1)$$

As we are interested in the case $l = 2$, four possible alternative theories can be set up:

$$L = \tfrac{1}{2}(m + n + 1) \qquad \begin{array}{ll} \text{1st:} & n = 0 \\ & m = 3 \\ \text{2nd:} & n = 1 \\ & m = 2 \\ \text{3rd:} & n = 2 \\ & m = 1 \\ \text{4th:} & n = 3 \\ & m = 0 \end{array}$$

Let us now calculate the values of (j, j') in each case (we number them, respectively, 1a, 2b, 3c, 4d):

1a
$$n = 0\ j_\xi = \tfrac{1}{2}(n + 1), \qquad j'_\xi = \tfrac{1}{2}m, \qquad j_\xi = \tfrac{1}{2}; j'_\xi = \tfrac{3}{2}$$
$$m = 3\ j_\zeta = \tfrac{1}{2}n, \qquad j'_\zeta = \tfrac{1}{2}(m + 1), \qquad j_\zeta = 0, \qquad j'_\zeta = 2$$

2b
$$j_\xi = \tfrac{1}{2}(n + 1) = 1, \qquad j'_\xi = \tfrac{1}{2}m = 1$$
$$j_\zeta = \tfrac{1}{2}n = \tfrac{1}{2}, \qquad j'_\zeta = \tfrac{1}{2}(m + 1) = \tfrac{3}{2}$$

3c
$$j_\xi = \tfrac{1}{2}(n + 1) = \tfrac{3}{2}, \qquad j'_\xi = \tfrac{1}{2}m = \tfrac{1}{2}$$
$$j_\zeta = \tfrac{1}{2}n = 1, \qquad j'_\zeta = \tfrac{1}{2}(m + 1) = 1$$

4d
$$j_\xi = \tfrac{1}{2}(n + 1) = 2, \qquad j'_\xi = \tfrac{1}{2}m = 0$$
$$j_\zeta = \tfrac{1}{2}n = \tfrac{3}{2} = , \qquad j'_\zeta = \tfrac{1}{2}(m + 1) = \tfrac{1}{2}$$

The corresponding representations for these theories are (designating as previously, adding a, b, c, d)

1aa
$$\Lambda^\dagger = \Lambda^{1/2} \otimes \Lambda^{3/2} \dotplus \Lambda^0 \otimes \Lambda^2 = \Lambda^{1/2 + 3/2} \dotplus \Lambda^{3/2 - 1/2} \dotplus \Lambda^{2 + 0} = \Lambda^\dagger = 2\Lambda^2 \dotplus \Lambda^1$$

2bb
$$\Lambda^\dagger = \Lambda^1 \otimes \Lambda^1 \dotplus \Lambda^{1/2} \otimes \Lambda^{3/2} = \Lambda^{1+1} \dotplus \Lambda^{1+1-1} \dotplus \Lambda^{1/2 + 3/2} \dotplus \Lambda^{3/2 - 1/2} \dotplus \Lambda^{1-1} = 2\Lambda^2 \dotplus \Lambda^1 \dotplus \Lambda^0$$

3cc
$$\Lambda^\dagger = \Lambda^{3/2} \otimes \Lambda^{1/2} \dotplus \Lambda^1 \otimes \Lambda^1 = \Lambda^{3/2 + 1/2} \dotplus \Lambda^{3/2 - 1/2} \dotplus \Lambda^{1+1} \dotplus \Lambda^{1-1} \dotplus \Lambda^{1+1-1} = 2\Lambda^2 \dotplus 2\Lambda^1 \dotplus \Lambda^0$$

4dd $$\Lambda^{\dagger} = \Lambda^{2} \otimes \Lambda^{0} \dotplus \Lambda^{3/2} \otimes \Lambda^{1/2} = \Lambda^{2+0} \dotplus \Lambda^{3/2+1/2} \dotplus \Lambda^{3/2-1/2} + \Lambda^{3/2+1/2} = 2\Lambda^{2} \dotplus \Lambda^{1}$$

As we have seen, the 1st and 4th theories do not show up scalar representations; this fact has, possibly, something to do with the extra degree of freedom present in generalized massive spin-2 field equations, it is an open question.

We shall, to end up this part, write the Dirac generalized equations for each theory:

11aa
$$\partial_{A\dot{B}}\xi^{A}_{\dot{B}_1\dot{B}_2\dot{B}_3} = \kappa\,\zeta_{\dot{B}\dot{B}_1\dot{B}_2\dot{B}_3}$$
$$\partial^{A\dot{B}}\zeta_{\dot{B}\dot{B}_1\dot{B}_2\dot{B}_3} = \kappa\,\zeta^{A}_{\dot{B}_1\dot{B}_2\dot{B}_3}$$

22bb
$$\partial_{A\dot{B}}\xi^{AA_1}_{\dot{B}_1\dot{B}_2} = \kappa\,\zeta^{AA_1}_{\dot{B}\dot{B}_1\dot{B}_2}$$
$$\partial^{A\dot{B}}\zeta^{A_1}_{\dot{B}\dot{B}_1\dot{B}_2} = \kappa\,\zeta^{AA_1}_{B_1B_2}$$

33cc
$$\partial_{A\dot{B}}\xi^{AA_1A_2}_{\dot{B}\dot{B}_1} = \kappa\,\zeta^{A_1A_2}_{\dot{B}\dot{B}_1}$$
$$\partial^{A\dot{B}}\zeta^{A_1A_2}_{B_1} = \kappa\,\xi^{AA_1A_2}_{\dot{B}_1}$$

44dd
$$\partial_{A\dot{B}}\xi^{AA_1A_2A_3} = \kappa\,\zeta^{A_1A_2A_3}_{\dot{B}}$$
$$\partial^{A\dot{B}}\zeta^{A_1A_2A_3}_{B} = \kappa\,\xi^{AA_1A_2A_3}$$

Reference

Nunes, J. B. (1973). *Massive Spin-Two Fields and General Relativity,"* Ph.D. Thesis, University of London, King's College.